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Ghost field realizations of the spinor $W_{2,s}$ strings based on the linear $W_{1,2,s}$ algebras*

Yuxiao Liu, Lijie Zhang and Jirong Ren

Institute of Theoretical Physics, Lanzhou University
Lanzhou 730000, P.R. China
E-mail: zhanglj02@st.lzu.edu.cn

Abstract: It has been shown that certain W algebras can be linearized by the inclusion of a spin-1 current. This provides a way of obtaining new realizations of the W algebras. In this paper, we investigate the new ghost field realizations of the $W_{2,s}(s = 3, 4)$ algebras, making use of the fact that these two algebras can be linearized. We then construct the nilpotent $BRST$ charges of the spinor non-critical $W_{2,s}$ strings with these new realizations.

Keywords: BRST Quantization, Conformal and W Symmetry, BRST Symmetry.

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Contents

1. Introduction **1**

2. The linear $W_{1,2,s}$ algebras and the new ghost realizations of the $W_{2,s}$ algebras **2**

3. Ghost field realizations of the spinor non-critical $W_{2,s}$ strings **5**

3.1 Ghost field realizations of the spinor non-critical $W_{2,3}$ strings 6

3.2 Ghost field realizations of the spinor non-critical $W_{2,4}$ strings 7

4. Conclusion **10**

1. Introduction

W algebra has received considerable attention and application since its discovery in the middle of the 1980's [1]. Much work has been carried out on the classification of it and the study of the W gravity and W string theories. Furthermore, it also appears in the quantum Hall effect, black holes, in lattice models of statistical mechanics at criticality, and in other physical models [2] and so on.

In all applications of W algebra, the investigation of the W string is more interesting and important. The $BRST$ method [3] has turned out to be the simplest way to study the critical and non-critical W string theories. The $BRST$ charge of W_3 string was first constructed in [4], and the detailed studies of it can be found in [4]–[6]. The scalar field realizations of $W_{2,s}$ strings, which is a natural generalization of the W_3 string, have been obtained for $s = 4, 5, 6, 7$ [6]–[8]. In ref. [9] we discovered the reason that the scalar $BRST$ charge is difficult to be generalized to a general W_N string. At the same time, we found the methods to construct the spinor field realizations of $W_{2,s}$ strings and W_N strings [9,10]. Subsequently, we studied the exact spinor field realizations of $W_{2,s}(s = 3, 4, 5, 6)$ strings and $W_N(N = 4, 5, 6)$ strings [9]–[11]. Recently, the authors have constructed the nilpotent $BRST$ charges of spinor non-critical $W_{2,s}(s = 3, 4)$ strings by taking into account the property of spinor field [12]. These results will be of importance for constructing super W strings, and they will provide the essential ingredients.

However, all of these theories about the $W_{2,s}$ strings mentioned above are based on the non-linear $W_{2,s}$ algebras. Because of the intrinsic nonlinearity of the $W_{2,s}$ algebras, their study is a more difficult task compared to linear algebras. Fortunately, it has been shown that certain W algebras can be linearized by the inclusion of a spin-1 current. Once the linear algebra is constructed, one could algorithmically reproduce the structure of the corresponding nonlinear algebra. This provides a way of obtaining new realizations of the

$W_{2,s}$ algebras. Such new realizations were constructed for the purpose of building the corresponding scalar $W_{2,s}$ strings [13].

Since up to now there is no work focused on the research of ghost field realizations of the spinor $W_{2,s}$ strings based on the linear $W_{1,2,s}$ algebras, we will construct the new nilpotent $BRST$ charges of spinor non-critical $W_{2,s}$ strings for the first time by using the linear bases of the $W_{1,2,s}$ algebras in this paper. To construct a non-critical $BRST$ charge one must first solve the forms of matter currents T and W determined by the $OPEs$ of TT, TW and WW. T and W here are constructed by the linear bases T_0, J_0, W_0 of the $W_{1,2,s}$ algebras, and these linear bases are constructed with the ghost fields. Then direct substitution of these results into $BRST$ charge leads to the grading spinor field realizations. Such constructions are discussed for $s = 3, 4$ in detail. For the case of critical realizations, the corresponding results can be obtained naturally when the terms of matter currents vanish. All these results will be of importance for embedding of the Virasoro string into the $W_{2,s}$ strings.

The present paper is organized as follows. In section 2, we construct the linear bases of the $W_{1,2,s}$ algebras and give the ghost field realizations of the $W_{2,s}$ algebras. Then in section 3, we build new $BRST$ charges of the spinor non-critical $W_{2,3}$ and $W_{2,4}$ strings by using the grading $BRST$ method. And finally, the paper ends with a brief conclusion.

2. The linear $W_{1,2,s}$ algebras and the new ghost realizations of the $W_{2,s}$ algebras

The $W_{2,s}$ algebras can be linearized by the inclusion of a spin-1 current [14]. The $OPEs$ of the spin-1 current with the spin-2 current T and spin-s current W of the $W_{2,s}$ algebras are uniquely determined by requiring that the Jacobi identities be satisfied. The bases T and W then can be constructed by the linear bases of the $W_{1,2,s}$ algebras:

$$\begin{aligned} T &= T_0 \\ W &= W_0 + W_R(J_0, T_0) \end{aligned} \tag{2.1}$$

where the currents T_0, W_0 and J_0 generate the $W_{1,2,s}$ algebras. And the linearized $W_{1,2,3}$ and $W_{1,2,4}$ algebras take the form:

$$\begin{aligned} T_0(z)T_0(\omega) &\sim \frac{C/2}{(z-w)^4} + \frac{2T_0(\omega)}{(z-\omega)^2} + \frac{\partial T_0(\omega)}{(z-\omega)}\,, \\ T_0(z)W_0(\omega) &\sim \frac{sW_0(\omega)}{(z-\omega)^2} + \frac{\partial W_0(\omega)}{(z-\omega)}, \\ T_0(z)J_0(\omega) &\sim \frac{C_1}{(z-w)^3} + \frac{J_0(\omega)}{(z-\omega)^2} + \frac{\partial J_0(\omega)}{(z-\omega)}, \\ J_0(z)J_0(\omega) &\sim \frac{-1}{(z-\omega)^2}, \\ J_0(z)W_0(\omega) &\sim \frac{hW_0(\omega)}{(z-\omega)}, \\ W_0(z)W_0(\omega) &\sim 0\,, \end{aligned} \tag{2.2}$$

where $s = 3$ and 4 respectively. The coefficients C, C_1 and h are given by

$$\begin{aligned}
C &= 50 + 24t^2 + \frac{24}{t^2}, \quad C_1 = -\sqrt{6}\left(t + \frac{1}{t}\right), \quad h = \sqrt{\frac{3}{2}}t, \qquad (s = 3) \\
C &= 86 + 30t^2 + \frac{60}{t^2}, \quad C_1 = -3t - \frac{4}{t}, \quad h = t, \qquad (s = 4).
\end{aligned} \tag{2.3}$$

One can realize the algebras given in eq. (2.2) by the two-spinor realizations in which the current W_0 is zero. However, it was shown that W_0 does not have to be zero [14], we can alternatively realize it in terms of the ghost field in this paper. To obtain the new realizations for the linearized $W_{1,2,s}$ algebras, we use the bosonic ghost fields (R, S) with spins $(s, 1-s)$ and (b_1, c_1) with spins $(k, 1-k)$ to construct the linear bases of them. Then the realizations for the $W_{1,2,3}$ and $W_{1,2,4}$ algebras are given by

$$\begin{aligned}
T_0 &= T_{\text{eff}} + T_g, \\
J_0 &= \rho RS + \lambda b_1 c_1, \\
W_0 &= R,
\end{aligned} \tag{2.4}$$

where

$$T_g = sR\partial S + (s-1)\partial RS + kb_1\partial c_1 + (k-1)\partial b_1 c_1 \tag{2.5}$$

and T_{eff} is an arbitrary effective energy-momentum tensor with central C_{eff}. By making use of the $OPEs$ $J_0(z)J_0(\omega)$ and $J_0(z)W_0(\omega)$ in (2.2), we can solve the coefficients ρ and λ. From the OPE relation of T_0 and J_0 and eq. (2.3), we determine the value of k. Substituting this value into eq. (2.5), we can determine the central charge C_g of T_g with its OPE. Since the central charge C for T_0 is $C = C_g + C_{\text{eff}}$, the value of C_{eff} can be obtained. All the coefficients are listed as follows:

(i) $s = 3$

$$\begin{aligned}
\rho &= \sqrt{\frac{3}{2}}t, \quad \lambda = \pm\sqrt{1 - \frac{3}{2}t^2}, \quad k = \frac{1}{2} \mp \frac{\sqrt{6 - 9t^2}}{2t}, \\
C_g &= 46 + \frac{18}{t^2}, \quad C_{\text{eff}} = 4 + \frac{6}{t^2} + 24t^2.
\end{aligned} \tag{2.6}$$

(ii) $s = 4$

$$\begin{aligned}
\rho &= -t, \quad \lambda = \pm\sqrt{1 - t^2}, \quad k = \frac{1}{2} \mp \frac{2\sqrt{1 - t^2}}{t}, \\
C_g &= 97 + \frac{48}{t^2}, \quad C_{\text{eff}} = -11 + \frac{12}{t^2} + 30t^2.
\end{aligned} \tag{2.7}$$

Now let us turn our attention to the study of the ghost field realizations of the $W_{2,s}$ algebras with linear bases of the $W_{1,2,s}$ algebras. We begin by reviewing the structures of the $W_{2,s}$ algebras in conformal language. The OPE $W(z)W(\omega)$ for $W_{2,3}$ algebra is given by [1]

$$\begin{aligned} W(z)W(\omega) \sim\ & \frac{C/3}{(z-w)^6} + \frac{2T}{(z-\omega)^4} + \frac{\partial T}{(z-\omega)^3} + \\ & + \frac{1}{(z-\omega)^2}(2\Theta\Lambda + \frac{3}{10}\partial^2 T) + \frac{1}{(z-\omega)}\left(\Theta\partial\Lambda + \frac{1}{15}\partial^3 T\right), \end{aligned} \tag{2.8}$$

where

$$\Theta = \frac{16}{22+5C}, \qquad \Lambda = T^2 - \frac{3}{10}\partial^2 T. \tag{2.9}$$

For the case $W_{2,4}$, the OPE $W(z)W(\omega)$ take the form [15]:

$$\begin{aligned} W(z)W(\omega) \sim\ & \left\{\frac{2T}{(z-w)^6} + \frac{\partial T}{(z-\omega)^5} + \frac{3}{10}\frac{\partial^2 T}{(z-\omega)^4} + \right. \\ & \left. + \frac{1}{15}\frac{\partial^3 T}{(z-\omega)^3} + \frac{1}{84}\frac{\partial^4 T}{(z-\omega)^2} + \frac{1}{560}\frac{\partial^5 T}{(z-\omega)}\right\} + \\ & + \sigma_1\left\{\frac{U}{(z-\omega)^4} + \frac{1}{2}\frac{\partial U}{(z-\omega)^3} + \frac{5}{36}\frac{\partial^2 U}{(z-\omega)^2} + \frac{1}{36}\frac{\partial^3 U}{(z-\omega)}\right\} + \\ & + \sigma_2\left\{\frac{W}{(z-\omega)^4} + \frac{1}{2}\frac{\partial W}{(z-\omega)^3} + \frac{5}{36}\frac{\partial^2 W}{(z-\omega)^2} + \frac{1}{36}\frac{\partial^3 W}{(z-\omega)}\right\} + \\ & + \sigma_3\left\{\frac{G}{(z-\omega)^2} + \frac{1}{2}\frac{\partial G}{(z-\omega)}\right\} + \sigma_4\left\{\frac{A}{(z-\omega)^2} + \frac{1}{2}\frac{\partial A}{(z-\omega)}\right\} + \\ & + \sigma_5\left\{\frac{B}{(z-\omega)^2} + \frac{1}{2}\frac{\partial B}{(z-\omega)}\right\} + \frac{C/4}{(z-\omega)^8}, \end{aligned} \tag{2.10}$$

where the composites U (spin 4), and G, A and B (all spin 6), are defined by

$$\begin{aligned} U &= (TT) - \frac{3}{10}\partial^2 T, \qquad G = (\partial^2 TT) - \partial(\partial TT) + \frac{2}{9}\partial^2(TT) - \frac{1}{42}\partial^4 T, \\ A &= (TU) - \frac{1}{6}\partial^2 U, \qquad B = (TW) - \frac{1}{6}\partial^2 W, \end{aligned} \tag{2.11}$$

with normal ordering of products of currents understood. The coefficients $\sigma_i (i = 1-5)$ are given by

$$\begin{aligned} \sigma_1 &= \frac{42}{5C+22}, \qquad \sigma_2 = \sqrt{\frac{54(C+24)(C^2-172C+196)}{(5C+22)(7C+68)(2C-1)}}, \\ \sigma_3 &= \frac{3(19C-524)}{10(7C+68)(2C-1)}, \qquad \sigma_4 = \frac{24(72C+13)}{(5C+22)(7C+68)(2C-1)}, \\ \sigma_5 &= \frac{28}{3(C+24)}\sigma_2. \end{aligned} \tag{2.12}$$

The forms of T and W can be constructed with the linear bases T_0, J_0 and W_0. First we can write down the most general possible structure of W, then the relations of above

$OPEs$ determine the coefficients of the terms in W, the explicit results turn out to be very simple as follows:

(i) s=3

$$\begin{aligned} T &= T_0\,, \\ W &= W_0 + \zeta_1\partial^2 J_0 + \zeta_2 \partial J_0 J_0 + \zeta_3 J_0^3 + \zeta_4 \partial T_0 + \zeta_5 T_0 J_0\,, \end{aligned} \tag{2.13}$$

where

$$\begin{aligned} \zeta_1 &= 6\zeta_0(2+3t^2+2t^4)\,, \qquad \zeta_2 = 12\sqrt{6}\zeta_0 t(1+t^2)\,, \\ \zeta_3 &= 8\zeta_0 t^2\,, \qquad \zeta_4 = 3\sqrt{6}\zeta_0 t(1+t^2)\,, \qquad \zeta_5 = 12\zeta_0 t^2\,, \\ \zeta_0 &= \frac{1}{6t\sqrt{-15-34t^2-15t^4}}\,. \end{aligned}$$

(ii) s=4

$$\begin{aligned} T &= T_0\,, \\ W &= W_0 + \eta_1\partial^3 J_0 + \eta_2\partial^2 J_0 J_0 + \eta_3(\partial J_0)^2 + \eta_4 \partial J_0 (J_0)^2 + \eta_5 (J_0)^4 + \\ &\quad + \eta_6 \partial^2 T_0 + \eta_7 (T_0)^2 + \eta_8 \partial T_0 J_0 + \eta_9 T_0 \partial J_0 + \eta_{10} T_0 (J_0)^2\,, \end{aligned} \tag{2.14}$$

where

$$\begin{aligned} \eta_1 &= \eta_0(1800+5562t^2+7744t^4+6167t^6+2631t^8+450t^{10})\,, \\ \eta_2 &= 12\eta_0 t(450+1278t^2+1429t^4+752t^6+150t^8)\,, \\ \eta_3 &= 6\eta_0 t(1050+2932t^2+3009t^4+1353t^6+225t^8)\,, \\ \eta_4 &= 12\eta_0 t^2(4+3t^2)(150+226t^2+75t^4)\,, \\ \eta_5 &= 6\eta_0 t^3(150+226t^2+75t^4)\,, \\ \eta_6 &= 3\eta_0 t(240+724t^2+865t^4+465t^6+90t^8)\,, \\ \eta_7 &= 6\eta_0 t^3(3+t^2)(32+27t^2)\,, \\ \eta_8 &= 12\eta_0 t^2(150+376t^2+301t^4+75t^6)\,, \\ \eta_9 &= 12\eta_0 t^2(300+602t^2+376t^4+75t^6)\,, \\ \eta_{10} &= 12\eta_0 t^3(150+226t^2+75t^4)\,, \\ \eta_0 &= -(36t+12t^3)^{-1}(504000+3037560t^2+7617488t^4+10300470t^6+ \\ &\qquad + 8109196t^8+3716751t^{10}+918585t^{12}+94500t^{14})^{-1/2}\,. \end{aligned}$$

3. Ghost field realizations of the spinor non-critical $W_{2,s}$ strings

In this section, we construct the explicit ghost field realizations of the spinor non-critical $W_{2,s}$ strings for $s=3$ and 4. Since the non-critical $W_{2,s}$ strings are the theories of $W_{2,s}$ gravity coupled to a matter system, we introduce the matter currents T_m, W_m for the $W_{2,s}^m$ algebras in the beginning. Then we introduce the (b,c) ghost system with spins $(2,-1)$ for the spin-2 current, and the (β,γ) with spins $(s,1-s)$ for the spin-s current. The ghost fields b,c,β,γ are all bosonic and commuting. For the Liouville sector, the spinor field realizations of the $W^L_{2,s}$ algebras were constructed in our work of ref. [12]. We can instead realize the $W^L_{2,s}$ algebras here by two pairs of bosonic ghost fields (b_2,c_2) and (b_3,c_3). The $BRST$ charge takes the form:

$$Q_B = Q_0 + Q_1\,, \tag{3.1}$$

$$Q_0 = \oint dz\, c(T_L + T_m + KT_{bc} + yT_{\beta\gamma})\,, \tag{3.2}$$

$$Q_1 = \oint dz\, \gamma F(b_2,c_2,\beta,\gamma,T_m,W_m)\,, \tag{3.3}$$

where K,y are pending constants. The matter currents T_m and W_m, which have spin 2 and s respectively, generate the $W^m_{2,s}$ algebras. The energy-momentum tensors in eq. (3.2) are given by

$$T_L = 2b_2\partial c_2 + \partial b_2 c_2 + 2b_3\partial c_3 + \partial b_3 c_3\,, \tag{3.4}$$

$$T_{bc} = 2b\partial c + \partial b c\,, \tag{3.5}$$

$$T_{\beta\gamma} = s\beta\partial\gamma + (s-1)\partial\beta\gamma\,, \tag{3.6}$$

The operator $F(b_2,c_2,\beta,\gamma,T_m,W_m)$ has spin s and ghost number zero. The nilpotent $BRST$ charge generalizes the one for scalar non-critical $W_{2,s}$ strings, and it is also graded with $Q_0^2 = Q_1^2 = \{Q_0,Q_1\} = 0$. It should be emphasized that the first condition is satisfied for any s automatically, we only need the other two conditions to determine y and the coefficients of the terms in $F(b_2,c_2,\beta,\gamma,T_m,W_m)$.

We now discuss the exact solutions of ghost field realizations of the spinor non-critical $W_{2,3}$ and $W_{2,4}$ strings respectively, making use of the grading $BRST$ method and the procedure mentioned above.

3.1 Ghost field realizations of the spinor non-critical $W_{2,3}$ strings

In this case, the $BRST$ operator Q_B takes the form of (3.1). Considering the most extensive combinations with correct spin and ghost number, we can construct $F(b_2,c_2,\beta,\gamma,T_m,W_m)$ in eq. (3.3) as following:

$$\begin{aligned} F = {} & f_1\beta^3\gamma^3 + f_2\partial\beta\beta\gamma^2 + f_3\partial\beta\partial\gamma + f_4\beta\partial^2\gamma + f_5 b_2^3 c_2^3 + f_6\partial b_2 b_2 c_2^2 + f_7 b_2^2\partial c_2 c_2 + f_8\partial b_2\partial c_2 + \\ & + f_9\partial^2 b_2 c_2 + f_{10} b_2\partial^2 c_2 + f_{11}\beta^2\gamma^2 b_2 c_2 + f_{12}\beta\gamma b_2^2 c_2^2 + f_{13}\partial\beta\gamma b_2 c_2 + f_{14}\beta\partial\gamma b_2 c_2 + \\ & + f_{15}\beta\gamma\partial b_2 c_2 + f_{16}\beta\gamma b_2\partial c_2 + f_{17}\beta\gamma T_m + f_{18} b_2 c_2 T_m + f_{19}\partial T_m + f_{20} W_m\,. \end{aligned} \tag{3.7}$$

Substituting this expression back into eq. (3.3) and using the nilpotency conditions, we can calculate the results of y and $f_i (i = 1,2,\ldots,10)$. They correspond to three sets of solutions, i.e.

(i) $y = 0$ and

$$f_i = 0 \quad (i = 5-12, 17-20)\,, \qquad f_{13} = f_{16}\,, \qquad f_{14} = 2f_{16}\,, \qquad f_{15} = f_{16}\,,$$

and $f_j (j = 1-4, 16)$ are arbitrary constants but do not vanish at the same time.

(ii) $y = 1$ and

$$\begin{aligned}
f_1 &= \frac{1}{20}(-2f_2 - f_{11})\,, \qquad f_3 = \frac{1}{180}(480f_2 - 60f_4 - 90f_{14} + 100f_{15} + 80f_{16} - \\
&\qquad\qquad\qquad\qquad\qquad - 3C_m f_{18} - 4C_m f_{19})\,, \\
f_5 &= \frac{1}{72900}(27060f_2 - 39360f_4 - 8280f_8 + 15480f_9 + 67500f_{11} - 24480f_{14} + \\
&\quad +46700f_{15} + 2260f_{16} - 591C_m f_{18} - 1148C_m f_{19})\,, \\
f_6 &= \frac{1}{6}(2f_8 - 10f_9 + 5f_{14} - 10f_{15})\,, \\
f_7 &= \frac{1}{2700}(5280f_2 - 7680f_4 - 2340f_8 - 1260f_9 + 1260f_{14} + 1100f_{15} - \\
&\qquad - 3620f_{16} - 33C_m f_{18} - 224C_m f_{19})\,, \\
f_{10} &= \frac{1}{900}(5280f_2 - 7680f_4 - 540f_8 - 1260f_9 - 990f_{14} + 1100f_{15} + 880f_{16} - \\
&\qquad - 33C_m f_{18} - 224C_m f_{19})\,, \\
f_{12} &= \frac{1}{1080}(-660f_2 + 960f_4 - 2700f_{11} + 720f_{14} - 1060f_{15} - 380f_{16} + \\
&\qquad + 21C_m f_{18} + 28C_m f_{19})\,, \\
f_{13} &= \frac{1}{540}(660f_2 - 960f_4 - 360f_{14} + 700f_{15} + 560f_{16} - 21C_m f_{18} - 28C_m f_{19})\,, \\
f_{17} &= \frac{1}{10}(-3f_{18} - 4f_{19})\,,
\end{aligned}$$

where $f_j (j = 2, 4, 8, 9, 11, 14, 15, 16, 18, 19, 20)$ are arbitrary constants but do not vanish at the same time. C_m is the central charge corresponding to the matter current T_m.

(iii) y is an arbitrary constant and

$$\begin{aligned}
f_i &= 0 \qquad (i = 5 - 12, 17 - 20)\,, \\
f_1 &= -\frac{2}{10}f_2\,, \qquad f_3 = \frac{39}{16}f_2\,, \qquad f_4 = \frac{11}{16}f_2\,, \\
f_{13} &= f_{16}\,, \qquad f_{14} = 2f_{16}\,, \qquad f_{15} = f_{16}\,,
\end{aligned}$$

where f_2 and f_{16} are arbitrary constants but do not vanish at the same time.

Substituting the realizations (2.13) for $W_{2,3}$, with T_0, J_0 and W_0 given by (2.4), (2.5), (2.6), we obtain the solutions for the ghost field realizations of the spinor non-critical $W_{2,3}$ string.

3.2 Ghost field realizations of the spinor non-critical $W_{2,4}$ strings

Similarly, for the case $s = 4$, Q_B also takes the form of (3.1) and $F(b_2, c_2, \beta, \gamma, T_m, W_m)$ can be expressed in the following form:

$$\begin{aligned}
F &= g_1\beta^4\gamma^4 + g_2\partial\beta\beta^2\gamma^3 + g_3\beta^3\partial\gamma\gamma^2 + g_4\partial^2\beta\beta\gamma^2 + g_5\beta^2\partial^2\gamma\gamma + g_6(\partial\beta)^2\gamma^2 + g_7\beta^2(\partial\gamma)^2 + \\
&\quad +g_8\partial\beta\beta\partial\gamma\gamma + g_9\partial^3\beta\gamma + g_{10}\beta\partial^3\gamma + g_{11}b_2^4c_2^4 + g_{12}\partial b_2 b_2^2 c_2^3 + g_{13}b_2^3\partial c_2 c_2^2 + g_{14}\partial^2 b_2 b_2 c_2^2 \\
&\quad +g_{15}b_2^2\partial^2 c_2 c_2 + g_{16}(\partial b_2)^2 c_2^2 + g_{17}b_2^2(\partial c_2)^2 + g_{18}\partial b_2 b_2 \partial c_2 c_2 + g_{19}\partial^3 b_2 c_2 + g_{20} b_2 \partial^3 c_2 +
\end{aligned}$$

$$
\begin{aligned}
&+g_{21}\beta^3\gamma^3 b_2c_2+g_{22}\beta^2\gamma^2 b_2^2c_2^2+g_{23}\beta\gamma b_2^3c_2^3+g_{24}\partial\beta\beta\gamma^2 b_2c_2+g_{25}\beta^2\partial\gamma\gamma b_2c_2+\\
&+g_{26}\beta^2\gamma^2\partial b_2c_2+g_{27}\beta^2\gamma^2 b_2\partial c_2+g_{28}\partial\beta\gamma b_2^2c_2^2+g_{29}\beta\partial\gamma b_2^2c_2^2+g_{30}\beta\gamma\partial b_2b_2c_2^2+\\
&+g_{31}\beta\gamma b_2^2\partial c_2c_2+g_{32}\partial\beta\partial\gamma b_2c_2+g_{33}\beta\gamma\partial b_2\partial c_2+g_{34}\partial\beta\gamma\partial b_2c_2+g_{35}\partial\beta\gamma b_2\partial c_2+\\
&+g_{36}\beta\partial\gamma\partial b_2c_2+g_{37}\beta\partial\gamma b_2\partial c_2+g_{38}\partial^2\beta\gamma b_2c_2+g_{39}\beta\partial^2\gamma b_2c_2+g_{40}\beta\gamma\partial^2 b_2c_2+\\
&+g_{41}\beta\gamma b_2\partial^2 c_2+g_{42}\beta^2\gamma^2 T_m+g_{43}\partial\beta\gamma T_m+g_{44}\beta\partial\gamma T_m+g_{45}\beta\gamma\partial T_m+g_{46}b_2^2c_2^2T_m+\\
&+g_{47}\partial b_2c_2T_m+g_{48}b_2\partial c_2T_m+g_{49}b_2c_2\partial T_m+\\
&+g_{50}\beta\gamma b_2c_2T_m+g_{51}T_m^2+g_{52}\partial^2T_m+g_{53}W_m\,. \qquad (3.8)
\end{aligned}
$$

There are three sets of solutions:

(i) $y=0$ and

$$
\begin{aligned}
g_i &= 0 \quad (i=11-23,39,42,46-53)\,,\\
g_{24} &= 2g_{26}\,,\quad g_{25}=3g_{26}\,,\quad g_{27}=g_{26}\,,\quad g_{28}=\frac{1}{2}g_{30}\,,\quad g_{29}=g_{30}\,,\quad g_{31}=g_{30}\,,\\
g_{32} &= 2(g_{34}-g_{40})\,,\quad g_{35}=g_{33}+g_{34}-2g_{40}\,,\quad g_{36}=2g_{40}\,,\quad g_{37}=2(g_{33}-g_{40})\,,\\
g_{38} &= g_{34}-g_{40}\,,\quad g_{41}=g_{33}-g_{40}\,,\quad g_{43}=g_{45}\,,\quad g_{44}=2g_{45}\,,
\end{aligned}
$$

where $g_j(j=1-10,26,30,33,34,40,45)$ are arbitrary constants but do not vanish at the same time.

(ii) $y=1$ and

$$
\begin{aligned}
g_1 &= \frac{1}{140}(-8g_2+6g_3-3g_{21})\,,\\
g_{10} &= \frac{1}{12}(28g_5-14g_7+3g_{39})\,,\\
g_9 &= \frac{1}{96}(-90g_4-108g_5+24g_7+30g_8-6g_{33}+10g_{36}+8g_{37}-6g_{39}-14g_{40}-\\
&\quad -10g_{41}+C_mg_{44}-2C_mg_{45})\,,\\
g_{12} &= \frac{1}{9720}(-258720g_2+194040g_3-20610g_4+99364g_5-108768g_6-92808g_7+\\
&\quad +43126g_8-8424g_{14}+23520g_{26}-23520g_{27}-3558g_{33}+6592g_{34}-\\
&\quad -6592g_{35}-13662g_{36}-1620g_{37}+918g_{39}+24290g_{40}+13390g_{41}-\\
&\quad -2940C_mg_{42}-824C_mg_{43}-927C_mg_{44}+2678C_mg_{45}+108C_mg_{47})\,,\\
g_{15} &= \frac{7}{6}(g_{37}-g_{39}-2g_{41})\,,\\
g_{18} &= \frac{7}{3}(-2g_{33}+g_{36}+g_{37}-2g_{39})\,,\\
g_{19} &= \frac{1}{18}(-6g_{14}+7g_{36}-7g_{39}-14g_{40})\,,\\
g_{24} &= \frac{1}{9}(-80g_2+60g_3-6g_4-36g_5-48g_6+18g_8+10g_{26}+8g_{27}+C_mg_{42})\,,\\
g_{25} &= \frac{1}{6}(88g_2-66g_3+48g_4-16g_5+24g_7-16g_8+10g_{26}+8g_{27}+C_mg_{42})\,,\\
g_{31} &= \frac{1}{3}(21g_{26}-21g_{27}+3g_{30}-3g_{33}+4g_{34}-4g_{35}-3g_{36}+3g_{37}+5g_{40}+g_{41})\,,
\end{aligned}
$$

$$
\begin{aligned}
g_{46} &= \frac{1}{54}(882g_{42} + 58g_{43} - 113g_{44} + 168g_{45} - 43g_{47} - 11g_{48} - 22g_{51} - 5C_m g_{51})\,,\\
g_{49} &= \frac{1}{18}(110g_{43} - 13g_{44} - 84g_{45} + 25g_{47} + 20g_{48} + 22g_{51} + 5C_m g_{51})\,,\\
g_{50} &= \frac{2}{3}(-21g_{42} - 4g_{43} + 3g_{44} - 2g_{45})\,,\\
g_{52} &= \frac{1}{12}(-22g_{43} + 11g_{44} - 5g_{47} - 4g_{48} - 8g_{51} - C_m g_{51})\,,
\end{aligned}
$$

where $g_j(j = 2-8,\ 14,\ 21,\ 26,\ 27,\ 30,\ 33-37,\ 39-45,\ 47,\ 48,\ 51,\ 53)$ are arbitrary constants but do not vanish at the same time. The other coefficients $g_i(i = 12,\ 13,\ 16,\ 17,\ 20,\ 22,\ 23,\ 28,\ 29,\ 32,\ 38)$ have a more verbose form which is similar to g_{12}, we don't list them here.

(iii) y is an arbitrary constant and

$$
\begin{aligned}
g_i &= 0 \qquad (i = 11-23, 39, 42, 46-53)\,,\\
g_1 &= \frac{289}{22330}(3g_4 + 2g_5 - g_8)\,,\\
g_2 &= \frac{1}{1276}(957g_3 - 289(3g_4 + 2g_5 - g_8))\,,\\
g_6 &= \frac{1}{7656}(7713g_4 + 38g_5 - 19g_8)\,,\\
g_7 &= \frac{1}{116}(57g_4 + 270g_5 - 19g_8)\,,\\
g_9 &= \frac{63}{232}(-3g_4 - 2g_5 + g_8)\,,\\
g_{10} &= \frac{133}{696}(-3g_4 - 2g_5 + g_8)\,,\\
g_{24} &= 2g_{26}\,, \qquad g_{25} = 3g_{26}\,, \qquad g_{27} = g_{26}\,, \qquad g_{28} = \frac{1}{2}g_{30}\,, \qquad g_{29} = g_{30}\,,\\
g_{31} &= g_{30}\,, \qquad g_{32} = 2(g_{34} - g_{40})\,, \qquad g_{35} = g_{33} + g_{34} - 2g_{40}\,,\\
g_{36} &= 2g_{40}\,, \qquad G_{37} = 2(g_{33} - g_{40})\,, \qquad g_{38} = g_{34} - g_{40}\,,\\
g_{41} &= g_{33} - g_{40}\,, \qquad g_{43} = g_{45}\,, \qquad g_{44} = 2g_{45}\,,
\end{aligned}
$$

where $g_j(j = 3,\ 4,\ 5,\ 8,\ 26,\ 30,\ 33,\ 34,\ 40,\ 45)$ are arbitrary constants but do not vanish at the same time.

Substituting the realizations (2.14) for $W_{2,4}$, with T_0, J_0 and W_0 given by (2.4), (2.5), (2.7), we obtain the solutions for the ghost field realizations of the spinor non-critical $W_{2,4}$ string.

Since the solutions of $F(b_2, c_2, \beta, \gamma, T_m, W_m)$ are of general type, the constructions of Q_1 are general naturally. Noting that the constructions of the energy-momentum tensors in Q_0 include all combinations of various fields, the realizations of $Q_B = Q_0 + Q_1$ that we have obtained are of the most general constructions. For the case of the spinor critical $W_{2,s}$ strings, the realizations can be obtained when the matter currents T_m and W_m in eqs. (3.2), (3.3) vanish. It is easy to see that the constructions become relatively simple and there are also three sets of solutions for each s. For the reason of space, We don't list them here.

Note that our new realizations, based on spinor $W_{2,s}$ strings, are very different from that of the scalar $W_{2,s}$ strings. The critical center charges C for scalar $W_{2,s}$ are fixed, for example $C = 100$ for $s = 3$ and $C = 176$ for $s = 4$. One usually need an exact center charge for the Liouville sector to obtain the realizations of certain scalar non-critical $W_{2,s}$ string. But for spinor $W_{2,s}$ strings, the center charges are arbitrary for both the critical and non-critical cases. The reason is that the condition $Q_0^2 = 0$ is satisfied automatically as a result of the property of spinor field.

4. Conclusion

In conclusion, we have constructed the new ghost field realizations of the $W_{2,3}$ and $W_{2,4}$ algebras, making use of the fact that the $W_{2,3}$ and $W_{2,4}$ algebras can be linearized through the addition of a spin-1 current. The spin-s current W_0 in the linearized $W_{1,2,s}$ algebras is null. This null current is identically zero in the case of the two-spinor realizations of the $W_{1,2,s}$ algebras [16]. However, W_0 can be non-zero and is realized by the bosonic ghost fields in this paper. In addition, the other two currents T_0 and J_0 are also realized by the bosonic ghost fields. Then the non-linear bases of $W_{2,s}$ have been constructed with these linear bases T_0, J_0 and W_0. Subsequently, we use these new realizations to build the nilpotent $BRST$ charges of spinor non-critical $W_{2,s}$ strings. The $BRST$ charge generalizes the one for scalar non-critical $W_{2,s}$ strings, and it is graded with $Q_0^2 = Q_1^2 = \{Q_0, Q_1\} = 0$. In particular, using the procedure mentioned above, we have already discussed the cases of $s = 3$ and 4 in detail. It is easy to see that these constructions are very standard, that is, there are three solutions for $s = 3$ and 4, respectively. It is worth to point out that the results of the non-critical $W_{2,s}(s = 3, 4)$ strings would turn to that of the corresponding critical strings when the matter currents vanish. Similarly, other field realizations of $W_{2,s}$ strings can be obtained with our method. And furthermore, such realizations with higher spin s will be expected to exist. In view of these points, it would be interesting to investigate the physical states, which are defined as the cohomology classes of a nilpotent $BRST$ operator Q_B. Moreover other implications, such as the construction of the W-gravity, can be studied also.

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